Dynamic modelling of structural joints by using FRF decoupling

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OUTLINE

- Introduction
  - Motivation
  - Basic concepts (Spatial, Modal and Response Models, FRF)
- FRF coupling in structural dynamics
- Identifying connection dynamics by FRF decoupling
- Example applications
  - Identification of contact parameters in spindle–holder–tool assemblies
  - Identification of contact parameters in beams connected with bolted joints
- Nonlinearity matrix concept

INTRODUCTION

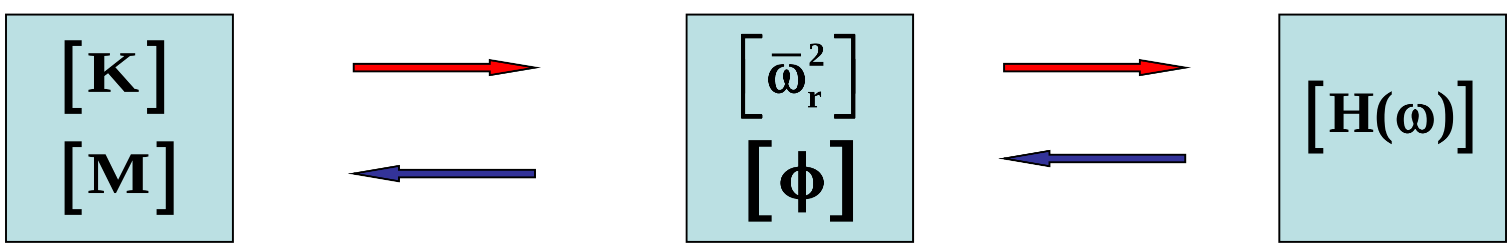
Motivation

- Mathematical modelling of structural joints is very important, especially in some applications such as aerospace structures
- Yet, it is still one of the challenging topics in structural dynamics
- In several applications theoretical approaches do not provide reliable mathematical models
- That makes it necessary to use ***experimental approaches to identify joint dynamic properties***
- Direct measurement of the interface is not possible without changing the interface
- Recent researches gave promising results to characterize contact interfaces by inverse methods based on experimental identification

Spatial, Modal and Response Models

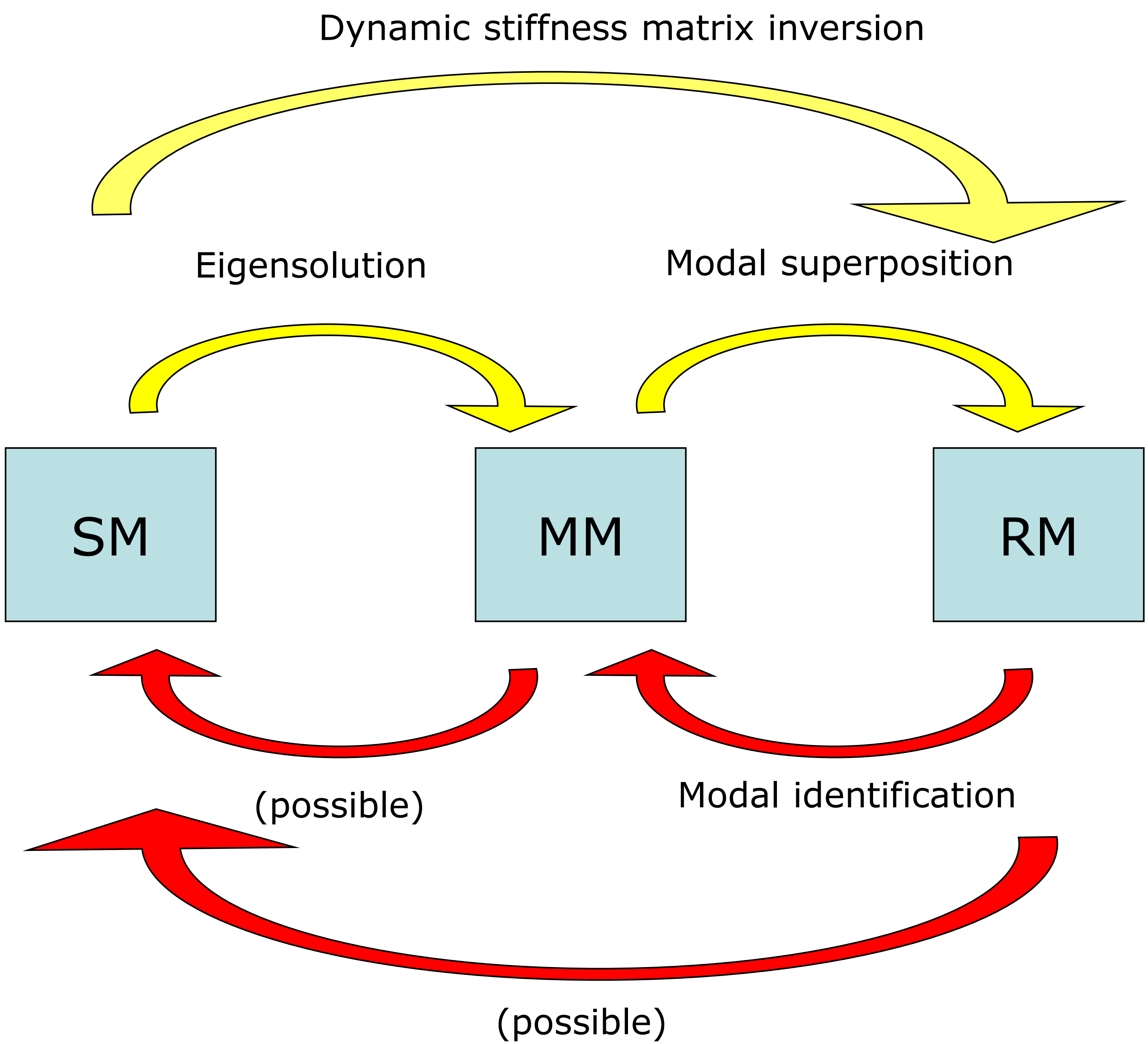
Description of structure	Vibration modes	Response
Mass Stiffness Damping	Natural frequencies Mode shapes Modal Damping	FRFs

Spatial model **Modal model** **Response model**



: Theoretical route

: Experimental route


Dynamic stiffness matrix inversion
Eigensolution
Modal superposition
SM
MM
RM
(possible)
Modal identification
(possible)

Frequency Response Functions (FRFs)

Receptance
(or Admittance, Dynamic Flexibility)

$$H(\omega) = \frac{x(t)}{f(t)}$$

Mobility

$$Y(\omega) = \frac{v(t)}{f(t)} = \frac{\dot{x}(t)}{f(t)}$$

Accelerance
(or *Inertance*)

$$A(\omega) = \frac{a(t)}{f(t)} = \frac{\ddot{x}(t)}{f(t)}$$

For harmonic excitation:

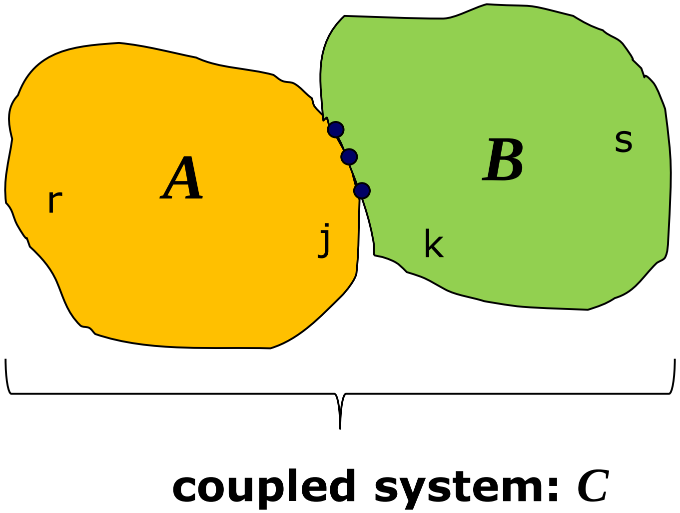
$$f(t) = F\, e^{i\omega t} \longrightarrow x(t) = X\, e^{i\omega t}$$

$$H(\omega) = \frac{x(t)}{f(t)} = \frac{X}{F}$$

FRF COUPLING

IN STRUCTURAL DYNAMICS

FRF coupling (Rigid coupling)



$[H_A]$: FRF matrix of subsystem A

$[H_B]$: FRF matrix of subsystem B

$[H_C]$: FRF matrix of coupled system

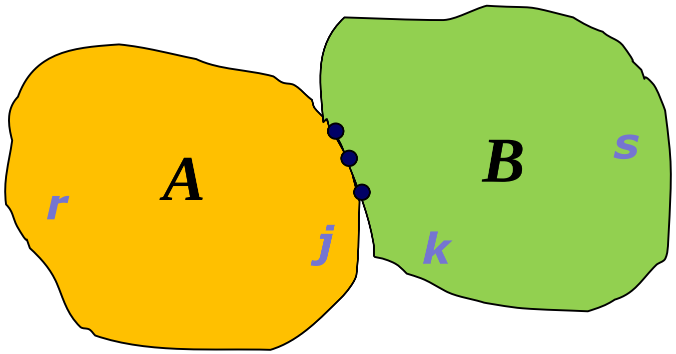
r : coordinates of subsystem A only

j : all connection coordinates of subsystem A

k : all connection coordinates of subsystem B

s : coordinates of subsystem B only

FRF coupling (Rigid coupling)



$$\{x_A\} = \begin{Bmatrix} \{x_r\} \\ \{x_j\} \end{Bmatrix} = [H_A]\{f_A\} = \begin{bmatrix} [H_{rr}] & [H_{rj}] \\ [H_{jr}] & [H_{jj}] \end{bmatrix} \begin{Bmatrix} \{f_r\} \\ \{f_j\} \end{Bmatrix}$$

$$\{x_B\} = \begin{Bmatrix} \{x_k\} \\ \{x_s\} \end{Bmatrix} = [H_B]\{f_B\} = \begin{bmatrix} [H_{kk}] & [H_{ks}] \\ [H_{sk}] & [H_{ss}] \end{bmatrix} \begin{Bmatrix} \{f_k\} \\ \{f_s\} \end{Bmatrix}$$

$$\{x_r\} = [H_{rr}]\{f_r\} + [H_{rj}]\{f_j\} \qquad (1)$$

$$\{x_j\} = [H_{jr}]\{f_r\} + [H_{jj}]\{f_j\} \qquad (2)$$

$$\{x_k\} = [H_{kk}]\{f_k\} + [H_{ks}]\{f_s\} \qquad (3)$$

$$\{x_s\} = [H_{sk}]\{f_k\} + [H_{ss}]\{f_s\} \qquad (4)$$

Assume no external force at connection DOFs:

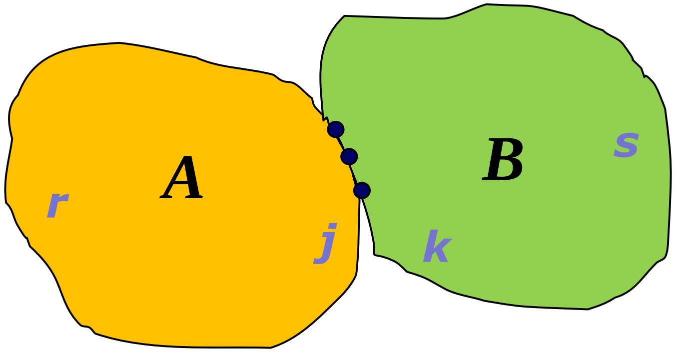
Equilibrium of internal forces:

$$\{f_j\} + \{f_k\} = 0$$

Compatibility of displacements:

$$\{x_j\} = \{x_k\}$$

FRF coupling (Rigid coupling)



Use force equilibrium and compatibility equations in (2) and (3) to eliminate connection forces in (1) and (4)

$$\{x_r\} = [H_{rr}]\{f_r\} + [H_{rj}]\{f_j\} \quad (1)$$

$$\{x_j\} = [H_{jr}]\{f_r\} + [H_{jj}]\{f_j\} \quad (2)$$

$$\{x_k\} = [H_{kk}]\{f_k\} + [H_{ks}]\{f_s\} \quad (3)$$

$$\{x_s\} = [H_{sk}]\{f_k\} + [H_{ss}]\{f_s\} \quad (4)$$

$$\{x_j\} = \{x_k\}$$

$$\{f_k\} = -\{f_j\}$$

FRF coupling (Rigid coupling)

$$\left[H_{jr}\right]\{f_r\}+\left[H_{jj}\right]\{f_j\}=-\left[H_{kk}\right]\{f_j\}+\left[H_{ks}\right]\{f_s\}$$

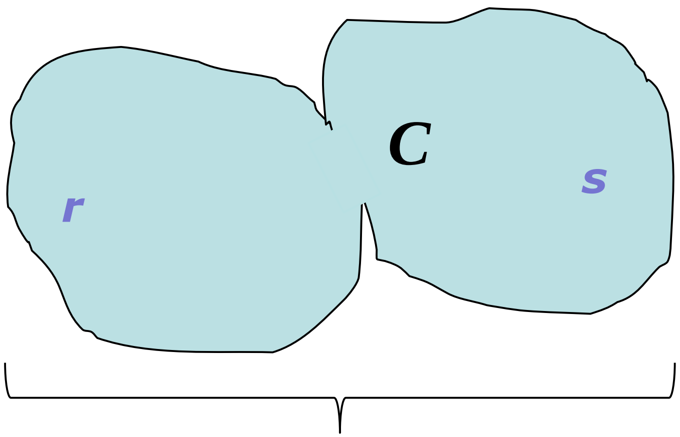
$$\{f_j\}=\left[H_{jj}+\left[H_{kk}\right]\right]^{-1}\left[H_{ks}\right]\{f_s\}-\left[H_{jj}+\left[H_{kk}\right]\right]^{-1}\left[H_{jr}\right]\{f_r\}$$

$$\{x_r\}=\left[H_{rr}\right]\{f_r\}+\left[H_{rj}\right]\{f_j\} \qquad (1)$$

$$\{x_s\}=\left[H_{sk}\right]\{f_k\}+\left[H_{ss}\right]\{f_s\} \qquad (4)$$

$$\{f_k\}=-\{f_j\}$$

FRF coupling (Rigid coupling)



coupled system

$\left[H_C\right]$: FRF matrix of coupled system

$$\begin{Bmatrix}\{x_r^C\}\\\{x_s^C\}\end{Bmatrix}=\left[H_C\right]\{f^C\}=\begin{bmatrix}\left[H_{rr}^C\right] & \left[H_{rs}^C\right]\\\left[H_{sr}^C\right] & \left[H_{ss}^C\right]\end{bmatrix}\begin{Bmatrix}\{f_r\}\\\{f_s\}\end{Bmatrix}$$

$\left[H_{rr}^C(\omega)\right]$

$$\{x_r^C\}=\left[\left[H_{rr}\right]-\left[H_{rj}\right]\left[\left[H_{jj}\right]+\left[H_{kk}\right]\right]^{-1}\left[H_{jr}\right]\right]\{f_r\}+$$
$$\left[H_{rj}\right]\left[\left[H_{jj}\right]+\left[H_{kk}\right]\right]^{-1}\left[H_{ks}\right]\{f_s\}$$

$\left[H_{rs}^C(\omega)\right]$

FRF coupling (Rigid coupling)

$$\left[H_{jr}\right]\left\{f_r\right\}+\left[H_{jj}\right]\left\{f_j\right\}=-\left[H_{kk}\right]\left\{f_j\right\}+\left[H_{ks}\right]\left\{f_s\right\}$$

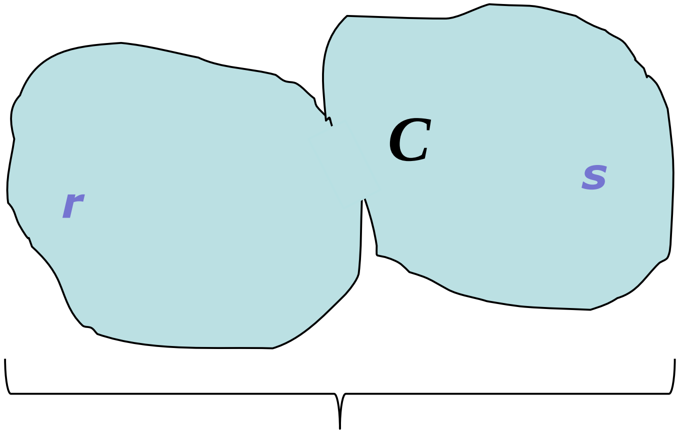
$$\left\{f_j\right\}=\left[H_{jj}+\left[H_{kk}\right]\right]^{-1}\left[H_{ks}\right]\left\{f_s\right\}-\left[H_{jj}+\left[H_{kk}\right]\right]^{-1}\left[H_{jr}\right]\left\{f_r\right\}$$

$$\left\{x_r\right\}=\left[H_{rr}\right]\left\{f_r\right\}+\left[H_{rj}\right]\left\{f_j\right\} \quad (1)$$

$$\left\{x_s\right\}=\left[H_{sk}\right]\left\{f_k\right\}+\left[H_{ss}\right]\left\{f_s\right\} \quad (4)$$

$$\left\{f_k\right\}=-\left\{f_j\right\}$$

FRF coupling (Rigid coupling)



$\left[H_C\right]$: FRF matrix of coupled system

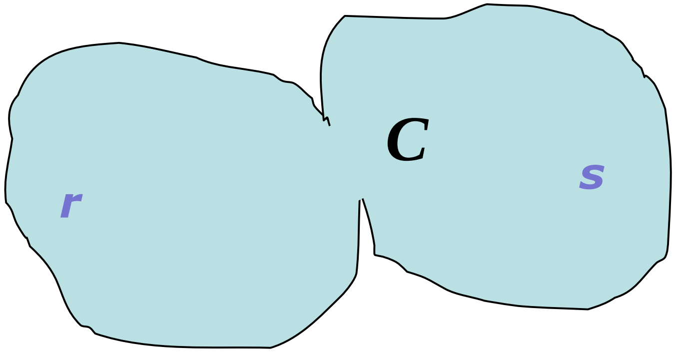
$$\begin{Bmatrix} \{x_r^C\} \\ \{x_s^C\} \end{Bmatrix} = \left[H_C\right]\left\{f^C\right\} = \begin{bmatrix} \left[H_{rr}^C\right] & \left[H_{rs}^C\right] \\ \left[H_{sr}^C\right] & \left[H_{ss}^C\right] \end{bmatrix} \begin{Bmatrix} \{f_r\} \\ \{f_s\} \end{Bmatrix}$$

$\left[H_{sr}^C\right]$

$$\left\{x_s^C\right\} = \left[H_{sk}\right]\left[\left[H_{jj}\right]+\left[H_{kk}\right]\right]^{-1}\left[H_{jr}\right]\left\{f_r\right\} + \left[\left[H_{ss}\right]-\left[H_{sk}\right]\left[\left[H_{jj}\right]+\left[H_{kk}\right]\right]^{-1}\left[H_{ks}\right]\right]\left\{f_s\right\}$$

$\left[H_{ss}^C\right]$

FRF coupling (Rigid coupling)



$$[H_C] = \begin{bmatrix} [H_{rr}^C] & [H_{rs}^C] \\ [H_{sr}^C] & [H_{ss}^C] \end{bmatrix}$$

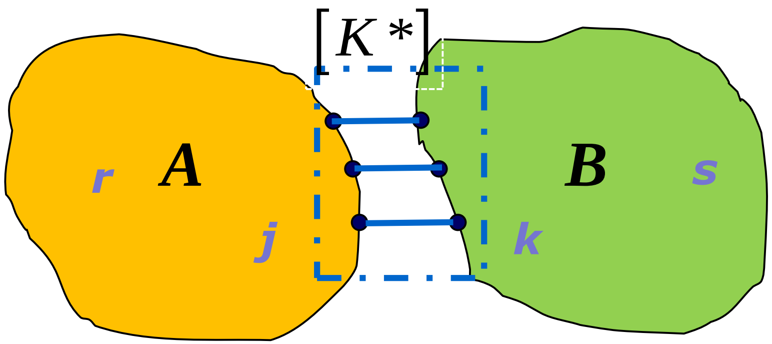
$$[H_{rr}^C] = [H_{rr}] - [H_{rj}]\left[[H_{jj}] + [H_{kk}]\right]^{-1}[H_{jr}]$$

$$[H_{rs}^C] = [H_{rj}]\left[[H_{jj}] + [H_{kk}]\right]^{-1}[H_{ks}]$$

$$[H_{sr}^C] = [H_{sk}]\left[[H_{jj}] + [H_{kk}]\right]^{-1}[H_{jr}]$$

$$[H_{ss}^C] = [H_{ss}] - [H_{sk}]\left[[H_{jj}] + [H_{kk}]\right]^{-1}[H_{ks}]$$

FRF coupling (Elastic coupling)



$$\{x_A\}=[H_A]\{f_A\}=\begin{bmatrix}[H_{rr}] & [H_{rj}]\\ [H_{jr}] & [H_{jj}]\end{bmatrix}\begin{Bmatrix}\{f_r\}\\ \{f_j\}\end{Bmatrix}$$

$$\{x_B\}=[H_B]\{f_B\}=\begin{bmatrix}[H_{kk}] & [H_{ks}]\\ [H_{sk}] & [H_{ss}]\end{bmatrix}\begin{Bmatrix}\{f_k\}\\ \{f_s\}\end{Bmatrix}$$

$$\{x_r\}=[H_{rr}]\{f_r\}+[H_{rj}]\{f_j\} \quad (1)$$

$$\{x_j\}=[H_{jr}]\{f_r\}+[H_{jj}]\{f_j\} \quad (2)$$

$$\{x_k\}=[H_{kk}]\{f_k\}+[H_{ks}]\{f_s\} \quad (3)$$

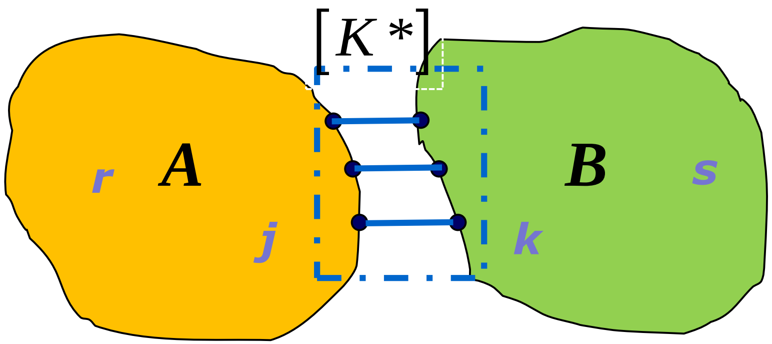
$$\{x_s\}=[H_{sk}]\{f_k\}+[H_{ss}]\{f_s\} \quad (4)$$

At connection DOFs:

$$\{f_j\}+\{f_k\}=0$$

$$[K^*]\{\{x_j\}-\{x_k\}\}=\{f_k\}$$

FRF coupling (Elastic coupling)



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$$\{x_B\} = [H_B]\{f_B\} = \begin{bmatrix} [H_{kk}] & [H_{ks}] \\ [H_{sk}] & [H_{ss}] \end{bmatrix} \begin{Bmatrix} \{f_k\} \\ \{f_s\} \end{Bmatrix}$$

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$$\{x_k\} = [H_{kk}]\{f_k\} + [H_{ks}]\{f_s\} \qquad (3)$$

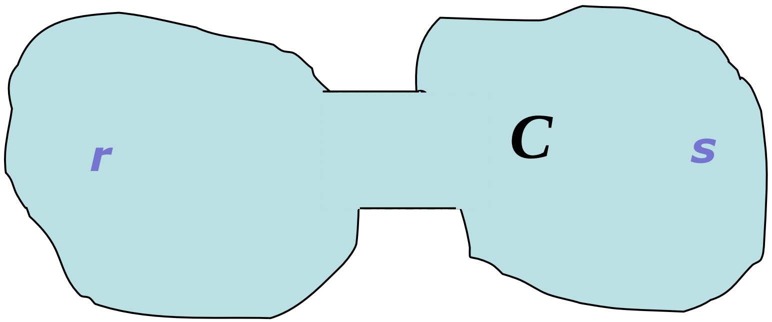
$$\{x_s\} = [H_{sk}]\{f_k\} + [H_{ss}]\{f_s\} \qquad (4)$$

At connection DOFs:

$$\{f_j\} + \{f_k\} = 0$$

$$[K^*]\{\{x_j\} - \{x_k\}\} = \{f_k\}$$

FRF coupling (Elastic coupling)



$$[H_C] = \begin{bmatrix} [H^C_{rr}] & [H^C_{rs}] \\ [H^C_{sr}] & [H^C_{ss}] \end{bmatrix}$$

$$[H^C_{rr}] = [H_{rr}] - [H_{rj}].\left[[H_{jj}] + [H_{kk}] + [K^*]^{-1}\right]^{-1}.[H_{jr}] \quad (5)$$

$$[H^C_{rs}] = [H_{rj}].\left[[H_{jj}] + [H_{kk}] + [K^*]^{-1}\right]^{-1}.[H_{ks}] \quad (6)$$

$$[H^C_{sr}] = [H_{sk}].\left[[H_{jj}] + [H_{kk}] + [K^*]^{-1}\right]^{-1}.[H_{jr}] \quad (7)$$

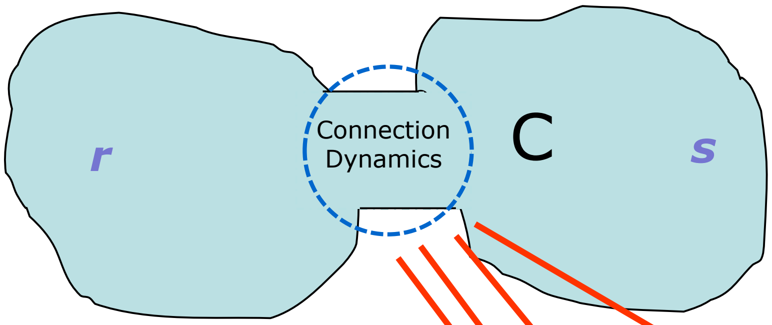
$$[H^C_{ss}] = [H_{ss}] - [H_{sk}].\left[[H_{jj}] + [H_{kk}] + [K^*]^{-1}\right]^{-1}.[H_{ks}] \quad (8)$$

Remarks

- In both formulations connection dofs are eliminated which keeps the matrix size small
- This is very useful especially when several substructures are assembled
- If FRF information at connection dofs of the coupled structure is required, the formulation must be slightly altered
- Coupling theoretically calculated FRFs is easier
- When experimental data is to be used, there will be difficulties in measuring receptances for rotational dofs

IDENTIFYING CONNECTION DYNAMICS BY FRF DECOUPLING

Identifying Connection Dynamics by FRF Decoupling



Recall elastic coupling equations:

$$\left[H_{rr}^{C}\right]=\left[H_{rr}\right]-\left[H_{rj}\right].\left[\left[H_{jj}\right]+\left[H_{kk}\right]+\left[K^{*}\right]^{-1}\right]^{-1}.\left[H_{jr}\right] \quad (5)$$

$$\left[H_{rs}^{C}\right]=\left[H_{rj}\right].\left[\left[H_{jj}\right]+\left[H_{kk}\right]+\left[K^{*}\right]^{-1}\right]^{-1}.\left[H_{ks}\right] \quad (6)$$

$$\left[H_{sr}^{C}\right]=\left[H_{sk}\right].\left[\left[H_{jj}\right]+\left[H_{kk}\right]+\left[K^{*}\right]^{-1}\right]^{-1}.\left[H_{jr}\right] \quad (7)$$

$$\left[H_{ss}^{C}\right]=\left[H_{ss}\right]-\left[H_{sk}\right].\left[\left[H_{jj}\right]+\left[H_{kk}\right]+\left[K^{*}\right]^{-1}\right]^{-1}.\left[H_{ks}\right] \quad (8)$$

Identifying Connection Dynamics by FRF Decoupling

Using equations (5) to (8), $[K^{*}]$ can be solved

$$[K*] = \left[[H_{jr}] . \left[[H_{rr}] - [H_{rr}^{C}] \right]^{-1} . [H_{rj}] - [H_{jj}] - [H_{kk}] \right]^{-1} \quad \text{(a)}$$

$$[K*] = \left[[H_{ks}] . [H_{rs}^{C}]^{-1} . [H_{rj}] - [H_{jj}] - [H_{kk}] \right]^{-1} \quad \text{(b)}$$

$$[K*] = \left[[H_{jr}] . [H_{sr}^{C}]^{-1} . [H_{sk}] - [H_{jj}] - [H_{kk}] \right]^{-1} \quad \text{(c)}$$

$$[K*] = \left[[H_{ks}] . \left[[H_{ss}] - [H_{ss}^{C}] \right]^{-1} . [H_{sk}] - [H_{jj}] - [H_{kk}] \right]^{-1} \quad \text{(d)}$$

Remarks

- **Theoretically**, equations (a) to (d) give **the same results**
- However, **due to using experimentally measured FRFs** (at least for the coupled system) each equation will yield **different result** – *which one is the best?*

Identifying Connection Dynamics by FRF Decoupling

- Equations are symmetric (a-d and b-c)
- Equation (a) or (d) may be be preferred

$$[K^*] = \left[\left[H_{jr}\right] . \left[\left[H_{rr}\right] - \left[H_{rr}^C\right] \right]^{-1} . \left[H_{rj}\right] - \left[H_{jj}\right] - \left[H_{kk}\right] \right]^{-1} \quad \text{(a)}$$

$$[K^*] = \left[\left[H_{ks}\right] . \left[H_{rs}^C\right]^{-1} . \left[H_{rj}\right] - \left[H_{jj}\right] - \left[H_{kk}\right] \right]^{-1} \quad \text{(b)}$$

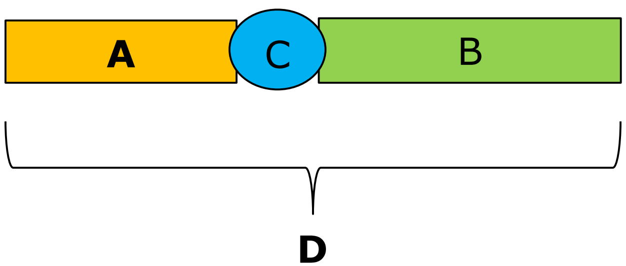
$$[K^*] = \left[\left[H_{jr}\right] . \left[H_{sr}^C\right]^{-1} . \left[H_{sk}\right] - \left[H_{jj}\right] - \left[H_{kk}\right] \right]^{-1} \quad \text{(c)}$$

$$[K^*] = \left[\left[H_{ks}\right] . \left[\left[H_{ss}\right] - \left[H_{ss}^C\right] \right]^{-1} . \left[H_{sk}\right] - \left[H_{jj}\right] - \left[H_{kk}\right] \right]^{-1} \quad \text{(d)}$$

Remarks

- **Theoretically**, equations (a) to (d) give **the same results**
- However, **due to using experimentally measured FRFs** (at least for the coupled system) each equation will yield **different result** – *which one is the best?*
- The sensitivity of each equation to measurement error may be different
- Decoupling is always problematic due to its nature – *a simple analogy*

A simple analogy



Coupling → Addition

A + B + C = D *Example*: **100+1+100 = 201**

± 1% error in **A**, **B** or **C** → error in **D** < ± 1%

Decoupling → Subtraction

D – (A + B) = C *Example:* **201–(100+ 100) = 1**

1% error in **D** → 200% error in **C**

At a different frequency it is possible to have **11 – (5 + 5) = 1**

1% error in **D** → 10% error in **C**

Remarks

- If connection dynamics is represented by *stiffness* and *viscous damping* elements: $K_{ij}^{*} = k_{ij} + i\omega c_{ij}$
- As an example, consider a beam connection where one translational and one rotational dofs are used:

$$[K^{*}] = \begin{bmatrix} (k_{yy} + i\omega c_{yy}) & (k_{y\theta} + i\omega c_{y\theta}) \\ (k_{\theta y} + i\omega c_{\theta y}) & (k_{\theta\theta} + i\omega c_{\theta\theta}) \end{bmatrix}$$

8 unknown stiffness and damping elements to be identified

- But, 8 equations will be written from each of Eqns (a) to (d), **at every frequency**

Remarks

- Even though one of the Eqns (a) to (d) is selected as the best one, still the equation can be used at any frequency
  - *Which frequency is the best?*
  - *Is it a good idea to take the average of the values obtained at each frequency?*
- Difficulty in measuring FRFs for rotational dofs
- Good news: ***No need to use FRFs at connection points*** (which are more difficult to measure)
- Various approaches were proposed to improve experimental substructure decoupling

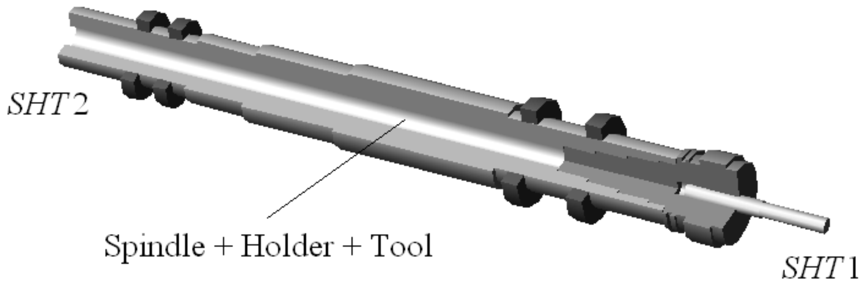
EXAMPLE APPLICATIONS

Identification of contact parameters in spindle–holder–tool assemblies

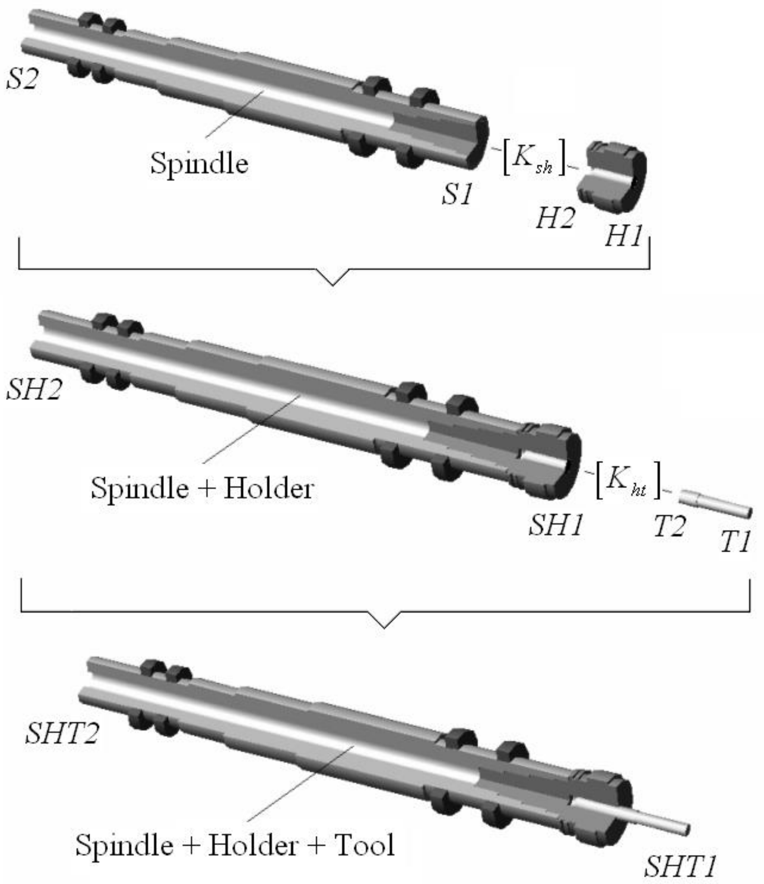
Objective:

To give an answer to the previous questions:

- Which frequency is the best for using the decoupling equations?
- Is it a good idea to take the average of the values calculated at several frequencies in a range?



Identification of contact parameters in spindle–holder–tool assemblies*



- Chatter stability regions in machining centers can be predicted if tool point FRFs are known
- In order to avoid experimental measurement for each combination of tool – holder, **mathematical models** have been developed
- For accurate prediction of tool point FRFs we need to model contact dynamics as well

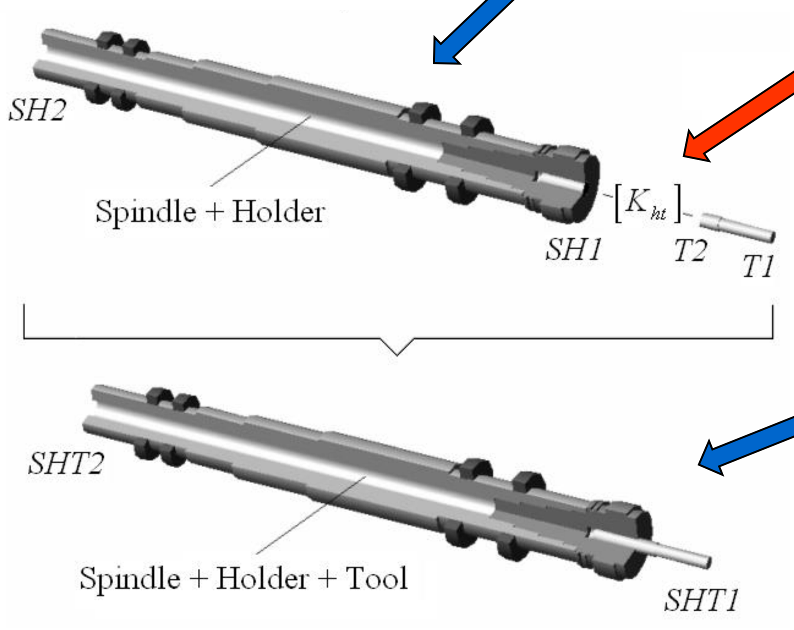
Özşahin, O., Ertürk, A, Özgüven, H. N. and Budak, E., "A Closed-Form Approach for Identification of Dynamical Contact Parameters in Spindle-Holder-Tool Assemblies", **International Journal of Machine Tools and Manufacture, v.49, pp. 25-35, 2009.*

Identification of contact parameters in spindle–holder–tool assemblies

Measured (S+H)

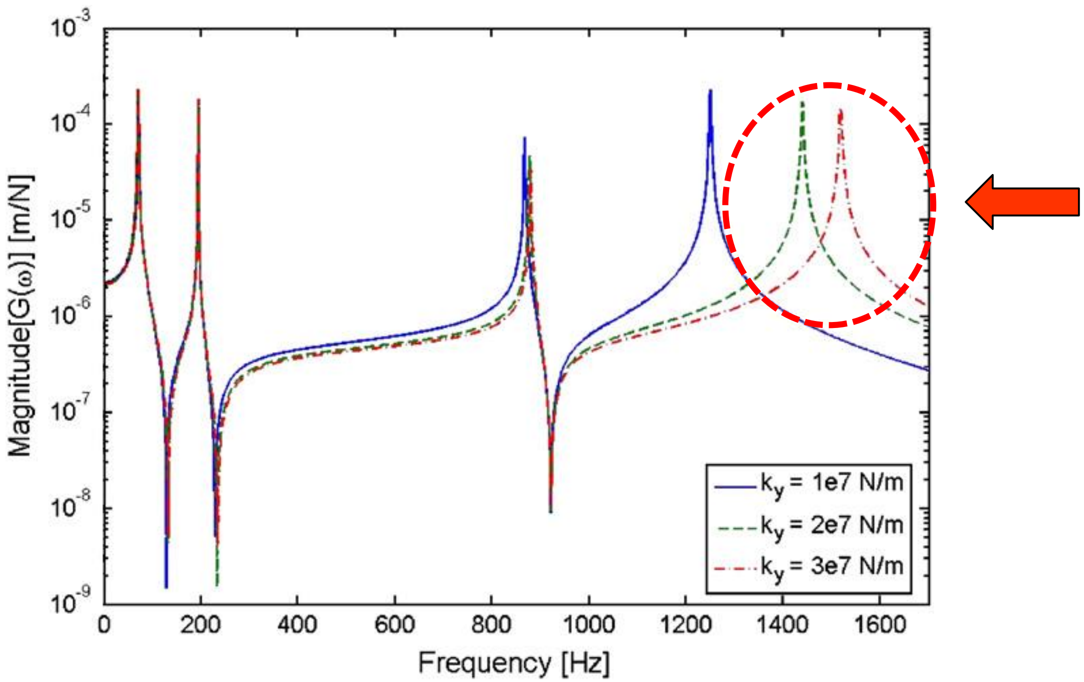
Identified (C)

Analytically determined (T)



Measured (S+H+C+T)

Identification of contact parameters in spindle–holder–tool assemblies



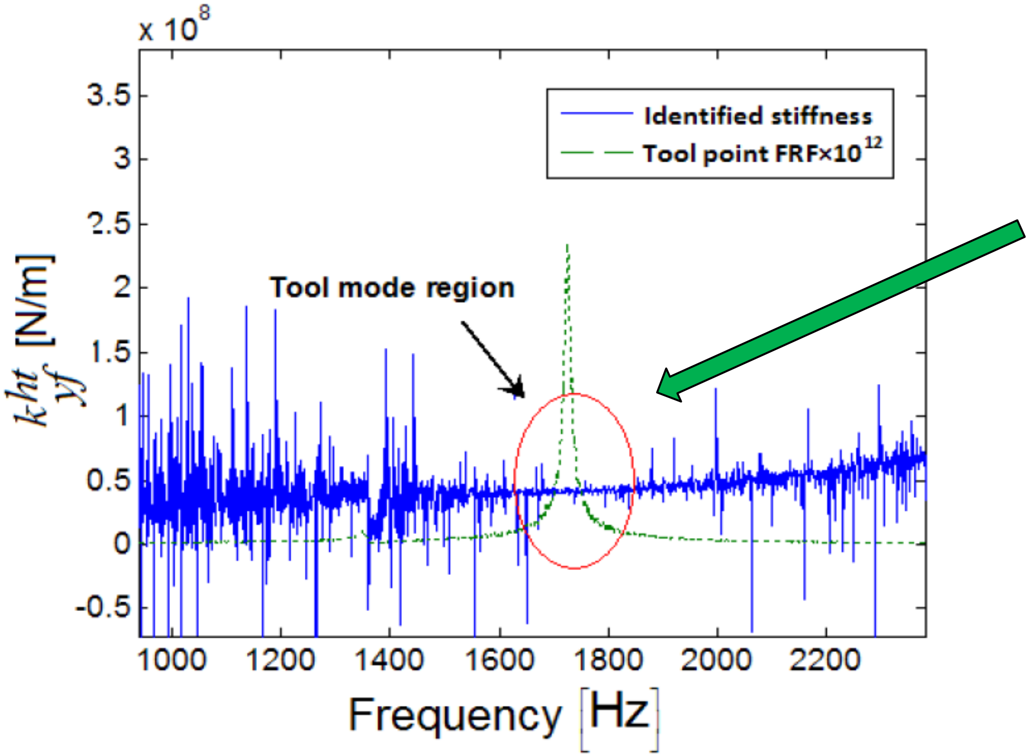
Sensitivity Analysis

By using the *mathematical model**, first determine *the mode at which connection dynamics have the maximum effect* on tool point FRF

Ertürk, A., Özgüven, H. N. and Budak, E., "Effect Analysis of Bearing and Interface Dynamics on Tool Point FRF for Chatter Stability in Machine Tools by using a New Analytical Model for Spindle-Tool Assemblies",* ***International Journal of Machine Tools and Manufacture**, v. 47, n.1, pp. 23-32, 2007.*

Identification of contact parameters in spindle–holder–tool assemblies

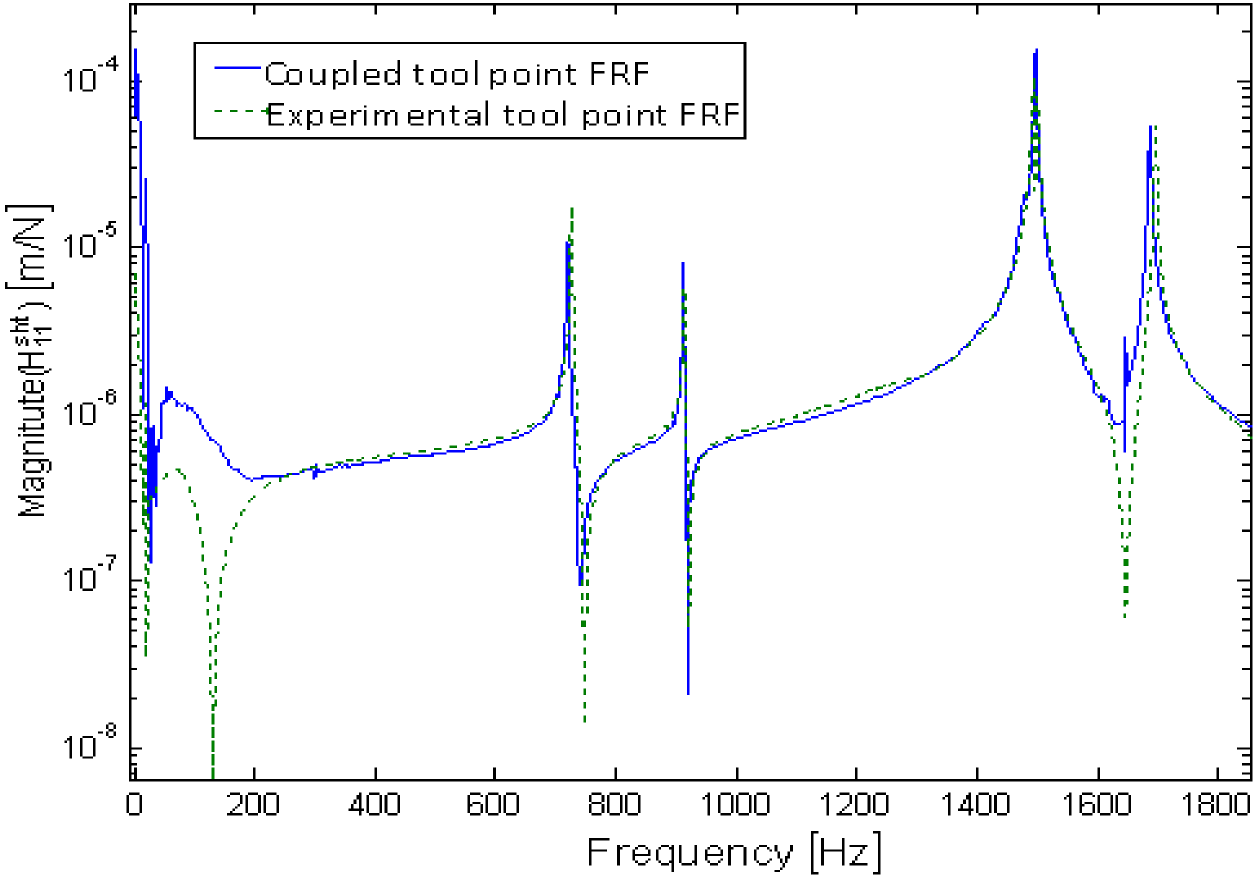
- Identify contact parameters by using FRF decoupling equation at each frequency in the whole range
- However, use only the values calculated at the mode where connection dynamics have the maximum effect on tool point FRF



- The identified parameters might have considerably different values even in this frequency range
- As a first attempt, the average values can be used

Identification of contact parameters in spindle–holder–tool assemblies

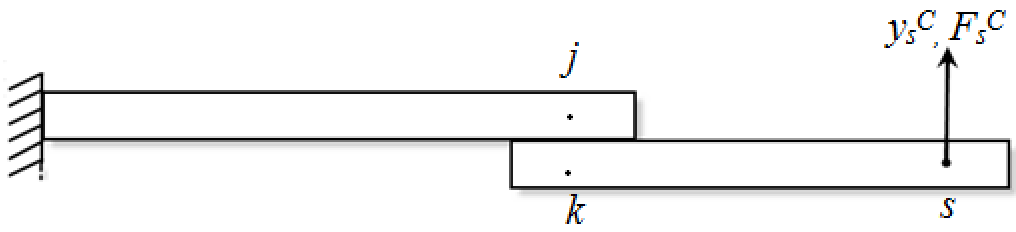
Experimental verification of the identified contact parameters



Identification of contact parameters in bolted joints

Objective:

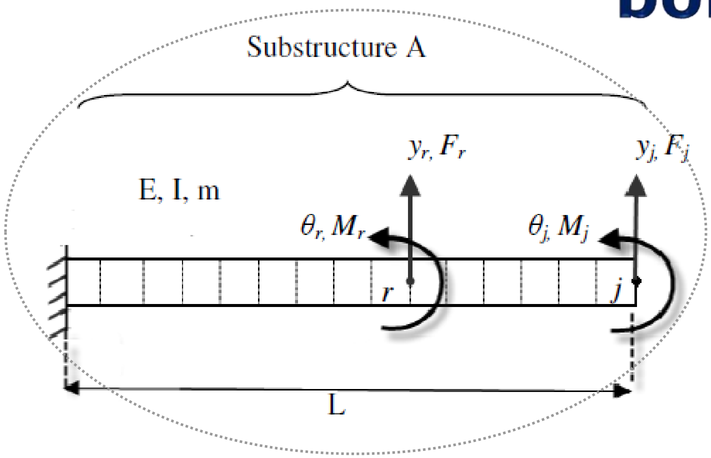
- To demonstrate the improvements in the joint identification method by employing optimization



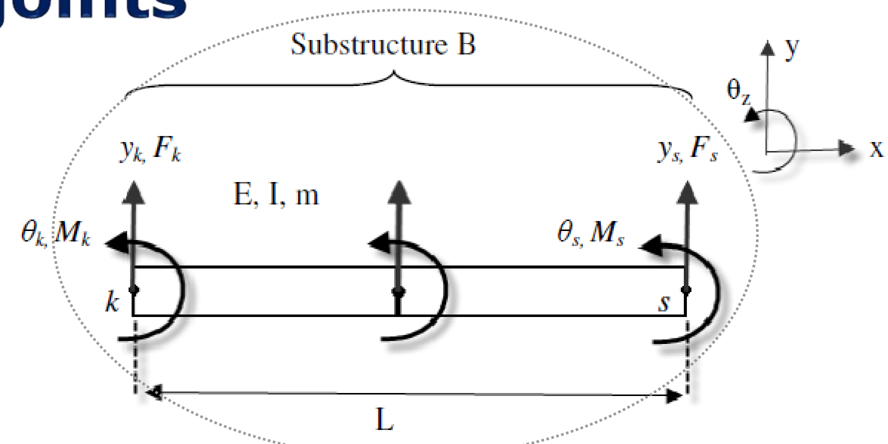
Identification of contact parameters in bolted joints

- Unmeasured FRFs (incomplete data in any of the FRF matrices we need to use) are estimated by using FRF synthesis
- FRFs for rotational dofs are estimated from measured FRFs for translational dofs
- Joint parameters identified by using *FRF decoupling* are improved by optimization
  - Joint stiffness
  - Joint damping

Identification of contact parameters in bolted joints*



Measured FRFs



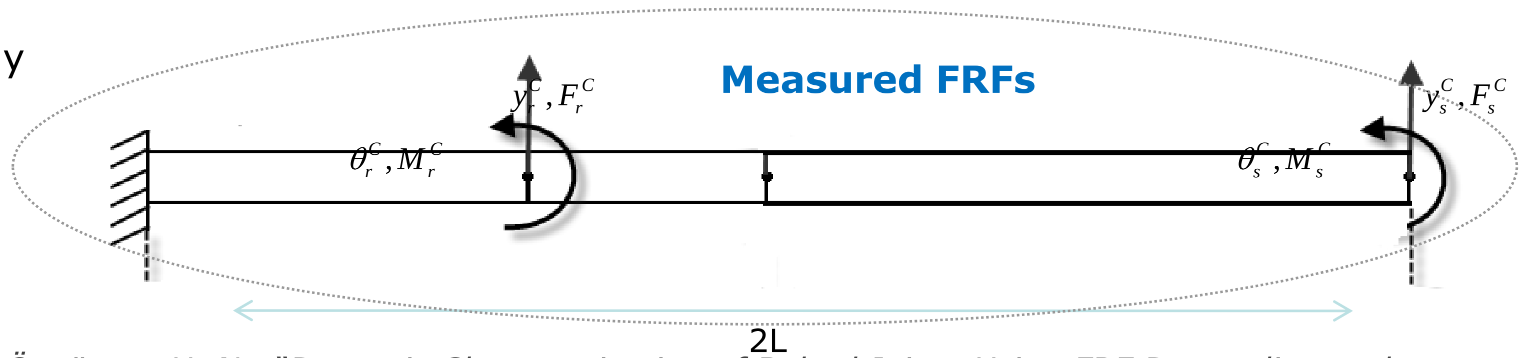
Theoretically obtained FRFs (FEM)

Joint Model:

$$[K^*] = \begin{bmatrix} (k_{yy} + j\omega c_{yy}) & (k_{y\theta} + j\omega c_{y\theta}) \\ (k_{\theta y} + j\omega c_{\theta y}) & (k_{\theta\theta} + j\omega c_{\theta\theta}) \end{bmatrix}$$

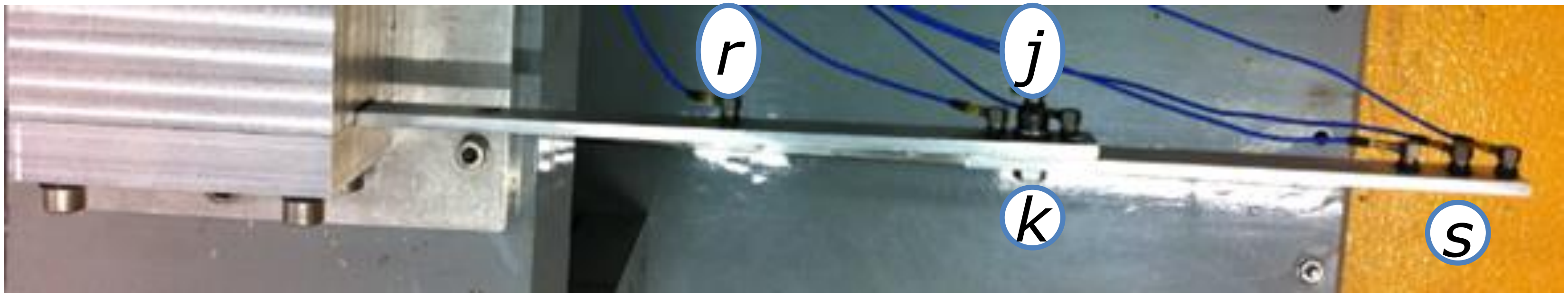
Identified

Assembly



*Tol, Ş, Özgüven, H. N., "Dynamic Characterization of Bolted Joints Using FRF Decoupling and Optimization", ***Mechanical Systems and Signal Processing***, v. 54-55, pp. 124-138, 2015

Identification of contact parameters in bolted joints*



Modeled with FEM

$$[K^*]=\left[[H_{ks}].\left[[H_{ss}]-[H_{ss}^{C}]\right]^{-1}.[H_{sk}]-[H_{jj}]-[H_{kk}]\right]^{-1}$$

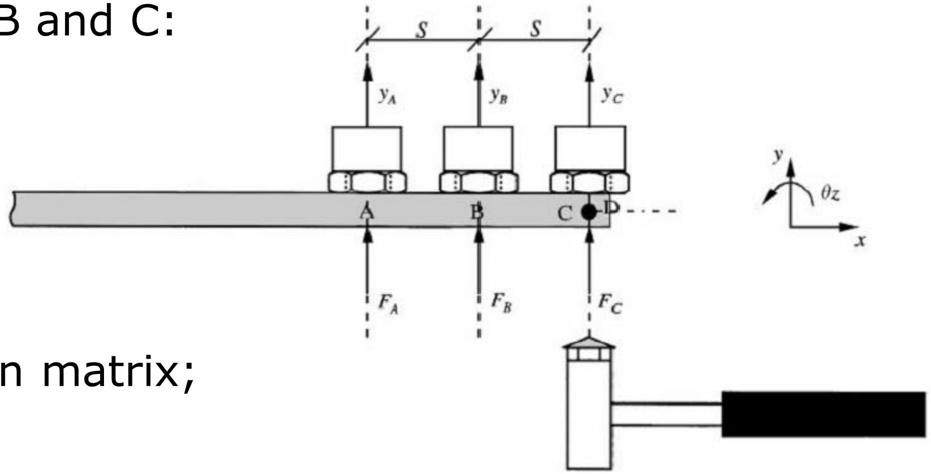
Identified

Measured

Identification of contact parameters in bolted joints

Estimation of FRFs for Rotational DOFs

Measured translational FRFs at points A, B and C:



$$[H_{meas}] = \begin{bmatrix} H_{AA} & H_{AB} & H_{AC} \\ H_{BA} & H_{BB} & H_{BC} \\ H_{CA} & H_{CB} & H_{CC} \end{bmatrix}$$

Using second-order-central transformation matrix;

$$[T_{2c}] = \frac{1}{2s}\begin{bmatrix} 0 & 2s & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

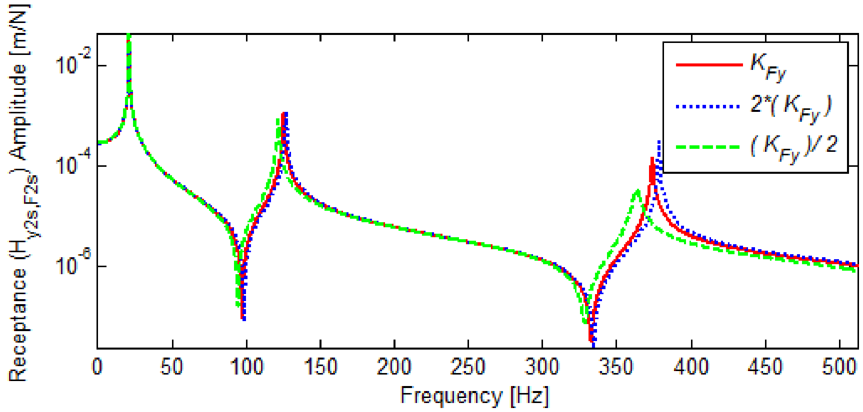
Estimated FRFs in y and θ directions at point B:

$$[H_{est}] = \begin{bmatrix} H_{yy} & H_{y\theta} \\ H_{\theta y} & H_{\theta\theta} \end{bmatrix} = [T_{2c}].[H_{meas}].[T_{2c}]^T$$

*Duarte, M.L.M., Ewins, D.J., "Rotational degrees of freedom for structural coupling analysis via finite difference technique with residual compensation", ***Mechanical Systems and Signal Processing***,v.14,pp.205-227-2000.

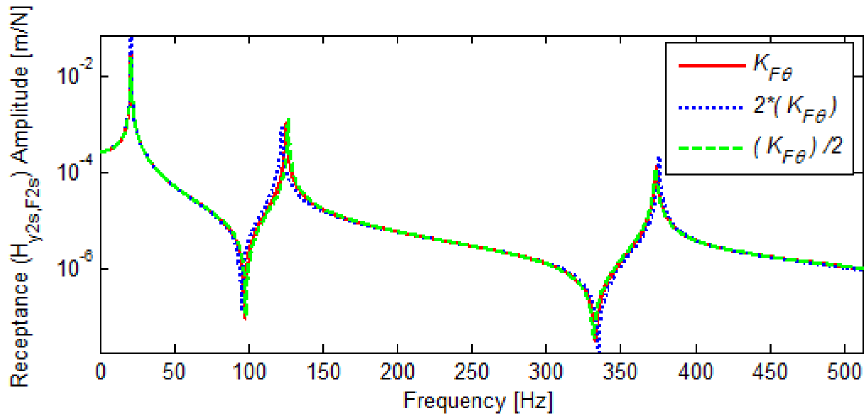
Identification of contact parameters in bolted joints

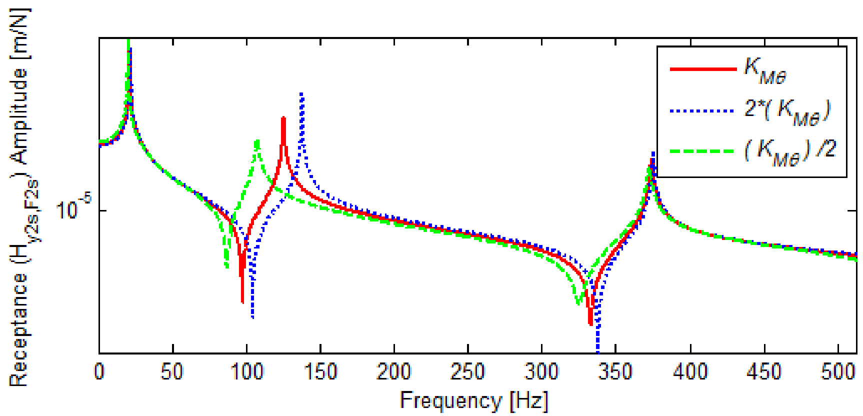
Sensitivity Analysis: Sensitivity of the coupled structure to connection dynamics



translational joint stiffness

cross-coupling joint stiffness



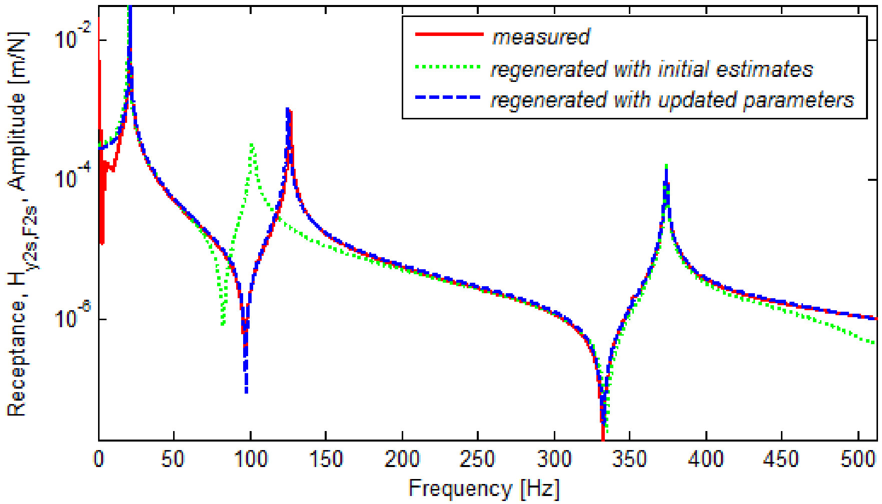


rotational joint stiffness.

Identification of contact parameters in bolted joints

- Improvement of the method by using optimization:
  - Joint properties are optimized such that regenerated and measured FRFs for the bolted assembly match at all the modes in the frequency range of interest
  - The key point for the success of the optimization process lies in starting with a good initial estimate
  - A set of joint parameters can be obtained starting with arbitrary initial estimates, and identified joint parameters may yield accurate FRF curves. However, when the same values are used in a new assembly, accurate FRFs cannot be obtained

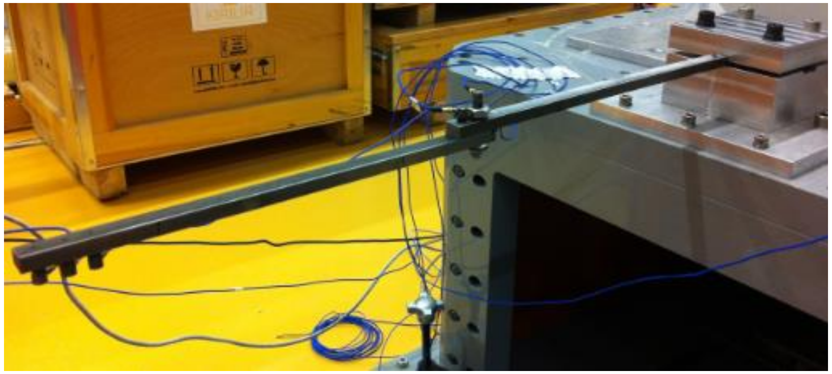
Identification of contact parameters in bolted joints

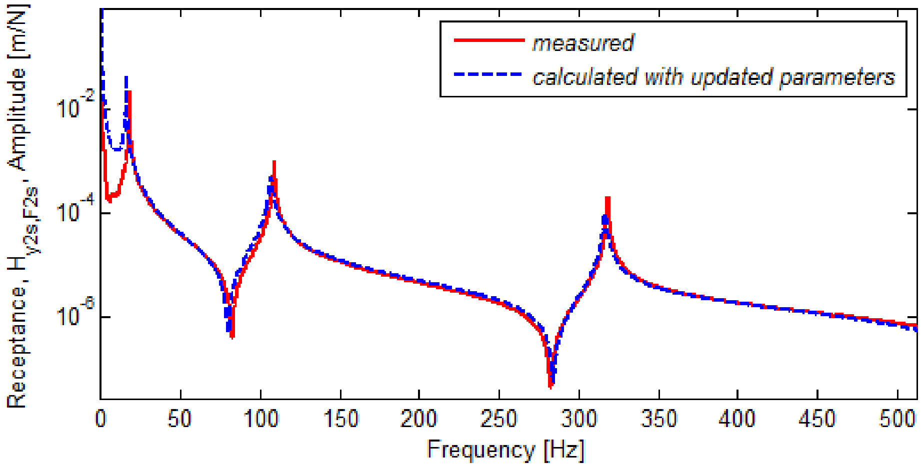


Regenerated FRF of the coupled structure using updated joint properties

Regenerated FRF of a new assembly by using updated joint properties

New assembly: Substructure B is longer





NONLINEARITY MATRIX

AND

DESCRIBING FUNCTION METHOD

Nonlinearity Matrix

Consider a nonlinear system under harmonic excitation

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+i[H]\{x\}+[K]\{x\}+\{N\}=\{f\} \qquad \text{where} \qquad \{f\}=\{F\}e^{i\omega t}$$

The response can be expressed in terms of Fourier series as

$$\{x\}=\sum_{m=0}^{\infty}\{x\}_m=\sum_{m=0}^{\infty}\{X\}_m e^{im\omega t}$$

The elements of the nonlinear internal force vector can be written as

$$N_r=\sum_{j=1}^{n} n_{rj} \qquad r=1,2,3\ldots n$$

where n_{rj} represents the nonlinear force:

between coordinates r and j for $j \neq r$
between r^{th} coordinate and ground for $j = r$

Nonlinearity Matrix

Considering only the first harmonic, the response can be written as

$$\{x\} = \{X\} e^{i\omega t}$$

Then, for symmetric nonlinearities (for simplicity), the nonlinear internal force components can be written as

$$n_{rj} = n_{rj}(x_{rj}) = \left(\frac{i}{\pi} \int_0^{2\pi} n_{rj}(x_{rj}) e^{-i\tau} d\tau \right) e^{i\omega t} \qquad (1)$$

where

$$x_{rj} = x_r - x_j \qquad for \quad j \neq r$$

$$x_{rj} = x_r \qquad for \quad j = r$$

Nonlinearity Matrix

The nonlinear internal forces, $n_{rj}(x_{rj})$ can also be expressed in terms of describing functions, v_{rj} as follows:

$$n_{rj}(x_{rj}) = v_{rj}\left(\left|X_{rj}\right|\right) X_{rj} e^{i\omega t} \qquad (2)$$

From (1) and (2)

$$v_{rj}\left(\left|X_{rj}\right|\right) = \frac{i}{\pi\left|X_{rj}\right|}\int_0^{2\pi} n_{rj}(x_{rj}) e^{-i\tau} d\tau$$

Then

$$N_r = \sum_{j=1}^{n} v_{rj} X_{rj} e^{i\omega t} \qquad (3)$$

On the other hand, the nonlinear internal force vector can be expressed as a multiplication of the so called "***nonlinearity matrix***" and displacement vector

$$\{N\} = [\Delta]\{X\} e^{i\omega t} \qquad (4)$$

From (3) and (4), the elements of [Δ] can be written as

$$\Delta_{kk} = v_{kk} + \sum_{\substack{j=1 \\ j\neq k}}^{n} v_{kj}$$

$$\Delta_{kj} = -v_{kj} \qquad for\ j \neq k$$

Harmonic Response in Nonlinear Structures by Using Nonlinearity Matrix*

Then, for the nonlinear system

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+i[H]\{x\}+[K]\{x\}+\{N\}=\{F\}e^{i\omega t}$$

$\{x\}=\{X\}e^{i\omega t}$ and $\{N\}=\{G\}e^{i\omega t}$

$\{G\}=[\Delta]\{X\}$ ⬅ **Key equation**

$[\Delta]$: **Nonlinearity matrix** (response level dependent)

$$\left[H^{NL}\right]=\left[-\omega^2[M]+i\omega[C]+i[H]+[K]+[\Delta]\right]^{-1}$$

Budak, E. and Özgüven, H. N. "Iterative Receptance Method for Determining Harmonic Response of Structures with Symmetrical Non-Linearities", **Mechanical Systems and Signal Processing, v.7, n.1, pp. 75-87, January 1993.*

Tanrıkulu, Ö., Kuran, B., Özgüven, H. N. and Imregün, M., "Forced Harmonic Response Analysis of Nonlinear Structures Using Describing Functions", **AIAA Journal, v.31, n.7, pp. 1313-1320, July 1993.*

Harmonic Response in Nonlinear Structures by Using Nonlinearity Matrix

$$\left[H^{NL}\right]=\left[-\omega^2[M]+i\omega[C]+i[H]+[K]+[\Delta]\right]^{-1}$$

Stiffness matrix → $[K]$

Nonlinearity matrix → $[\Delta]$

(acts as a *response dependent equivalent stiffness matrix*)

For nonlinear analysis:

$[H]$ ➡ Calculate by using modal analysis

$\left[H^{NL}\right]$ ➡ Calculate by using structural modification method using $[H]$ and $[\Delta]$

Solution requires iteration: $\{X_i\}=\left[H_i^{NL}\right]\{F\}$ ➡ $\left[\Delta_{i+1}\right]$ ➡ $\left[H_{i+1}^{NL}\right]$

Remarks

- Nonlinearity matrix acts like an equivalent complex stiffness matrix
- But it is response level dependent, and therefore solution requires iterations
- Formulation given here is for single harmonic analysis, however ***it can be extended for multi-harmonic analysis***
- Nonlinearity matrix concept is very useful and it can be used in ***extending several structural dynamics methods applicable to linear structures to nonlinear structures***:
  - *Nonlinear harmonic response analysis*
  - *Nonlinear structural modification analysis*
  - *Nonlinear structural coupling analysis*
  - *Nonlinear identification*
  - *Nonlinear model updating*

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https://blog.metu.edu.tr/ozguven/en